Mathematical Proofs

Universal Quantifier: This is the symbol $\forall$ and we use it when we want to indicate that a statement holds **for all** elements x in some set U. In this case we write $\forall x \in U,\ P(x)$, where $P(x)$ represents the statement that is true. For example, we might write $\forall x \in \mathbb{R},\ (-x)^2 = x^2$.

Existential Quantifier: This is the symbol $\exists$ and we use it when we want to indicate that a statement holds for **at least one** element x in some set U. In this case we write $\exists x \in U, P(x)$. For example, we might write $\exists x \in \mathbb{N},\ x + 2 = 4$. Note that using the symbol $\exists$ does not imply that there is only one such element, only that there is at least one such element.

Types of Proof

1. **Proof by Cases:**
   This sort of proof is usually used with the existential quantifier $\exists$ but can also be used occasionally with the universal quantifier $\forall$ if the set U is small enough.

   Example: Show there exists $a, b \in \mathbb{R}\backslash\mathbb{Q}$ (i.e., a and b are irrational numbers) such that $a^b \in \mathbb{Q}$.

   Solution: Here we are being asked to find two irrational numbers a and b such that a raised to the power b is rational. The simplest examples of irrational numbers are the square roots of non-perfect squares, so let us try to find an example using these. If $a = b = \sqrt{2}$, then $a^b = \sqrt{2}^{\sqrt{2}}$. There are now two possibilities to consider. Firstly there is the possibility that $\sqrt{2}^{\sqrt{2}}$ is already rational in which case we have our example. On the other hand it may be the case that $\sqrt{2}^{\sqrt{2}}$ is irrational. In this case we can take $a = \sqrt{2}^{\sqrt{2}}$ (since it is irrational) and $b = \sqrt{2}$. However then $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$ which is rational, so that we have our example. Notice that in this proof we have shown that such an a and b must exist but we do not know what they are - we do not know whether the first case works or the second case works but we do know that at least one of them must!

2. **Proof by Counterexample:**
   This is usually used to show that a statement involving the universal quantifier $\forall$ is false.

   Example: Show that the statement $\forall x, y \in \mathbb{R},\ 2^x 2^y = 2^{xy}$ is false.

   Solution: Here we are being asked to find two real numbers x and y such that $2^x 2^y \neq 2^{xy}$. Let us take $x = 2$ and $y = 3$. Then $2^x 2^y = 2^2 2^3 = 2^5 = 32$ while $2^{xy} = 2^{2\times 3} = 2^6 = 64$. Since $32 \neq 64$, we have our counterexample.

   Example: Either prove the statement $\forall x, y \in \mathbb{R}^+,\ \log_2 x + \log_2 y = \log_2(x + y)$ or show that it is false.

   Solution: In this case we are not told if the statement is true or false so we have to start by deciding if it is true of not. After trying some simple examples it becomes clear that the statement is not true. For example let $x = 1$ and $y = 2$. Then $\log_2 x + \log_2 y = 0 + 1 = 1$ while $\log_2(x + y) = \log_2 3 \neq 1$, so we have our counterexample.

3. **Proof by Contraposition:**
   In this type of proof we begin by writing the statement to be proved in a different form which hopefully will be easier to prove than the original. If we start with the statement $\forall x \in U$, if $P(x)$ then $Q(x)$, we start by writing $\forall x \in U$, if $\neg Q(x)$ then $\neg P(x)$, where $\neg P(x)$ means the negation of $P(x)$ and similarly for $\neg Q(x)$.

   Example: Prove that for all integers n, if n^2 is even then n is even.

   Solution: We will prove the contrapositive of the original statement. That is we have to show for all integers n, if n is odd, then n^2 is odd. However if n is odd then $n = 2k+1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(k^2+2k)+1$, so we see that n^2 is also odd. Hence the contrapositive statement is true and so the original statement is also true.

4. **Proof by Induction:**
   Induction is used to prove a statement holds for all natural numbers bigger than or equal to a given natural number a (which will often be 0 or 1).

   In the standard form of induction (there is another form called **strong induction** which is not covered in this sheet) we have to prove two things; we first have to prove that the statement holds for this a and then we have to prove that if it holds for some natural number $k \geqslant a$, then it also holds for $k+1$.

   Example: Prove that 9 divides $4^n + 6n - 1$ for all natural numbers n.

   Solution: We first have to prove it true for $n = 1$. However $4^1 + 6 \times 1 - 1 = 9$, which is divisible by 9.
   Next we assume that $4^k + 6k - 1$ is divisible by 9 for some natural number k and try to prove that $4^{k+1} + 6(k+1) - 1$ is divisible by 9 (note we use k here rather than n since using n amounts to assuming what we are trying to prove).
   Now, since (by assumption) $4^k + 6k - 1$ is divisible by 9, we can write $4^k + 6k - 1 = 9m$ for some integer m. Then $4^{k+1} + 6(k+1) - 1 = 4\left(4^k + 6k - 1\right) - 18k + 9 = 4 \times 9m - 18k + 9 = 9\,(4m - 2k + 1)$, so that $4^{k+1} + 6(k+1) - 1$ is divisible by 9 as we wanted.
   Hence, by Mathematical Induction, 9 divides $4^n + 6n - 1$ for all natural numbers n.

   Example: Prove that $\displaystyle\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for all natural numbers n.

   Solution: We first note that $\displaystyle\sum_{j=1}^{1} \frac{1}{j(j+1)} = \frac{1}{2}$ and $\dfrac{1}{1+1} = \dfrac{1}{2}$, so it is true for $n = 1$.
   Next, we assume it is true for $n = k \in \mathbb{N}$, i.e., we assume

$$\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}. \tag{1}$$

   However then

$$\begin{aligned}\sum_{j=1}^{k+1} \frac{1}{j(j+1)} &= \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{by (1)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},\end{aligned}$$

   as we want, so if the sum is true for $n = k$, it is also true for $n = k+1$.
   Hence, by Mathematical Induction, the sum is true for all $n \in \mathbb{N}$.